Math and Fire

Lee Green PE
Mark Goodson PE

Without a doubt, most fire investigators have never heard of the Napierian logarithm. However, almost every fire investigator has witnessed the manifestations of this logarithm at work. The Napierian logarithm is referred to as "ln", and in a formula, would be written out as ln x or ln (x). The Napierian logarithm is also known as the natural log or natural logarithm.

Similarly, many fire investigators are familiar conceptually with Fourier's Law of Heat Transfer. But how does one actually use it? And finally, we tie in mathematically the concept of the time constant. The Napierian logarithm and the time constant are the mathematical tools used to work with not only Fourier's Law, but also problems involving voltages and pressures.

This article will define the natural logarithm, show how basic math is performed using the natural logarithm and show how the natural logarithm function math correlates to experiments and common sense that most fire investigators have performed or experienced.

What is a logarithm?

A logarithm is engineering (or mathematical) shorthand that can be explained by two numbers, 1 and 1000. In what we shall call "base 10" arithmetic, the logarithm of 1 is 0. The logarithm of 1000 is 3. The number (0 or 3) simply denotes how many zeroes are placed behind the number "1." Here is a chart of some "base 10" logarithms:

Number	Logarithm
1	0
10	
100	2
1000	3
10000	4
100000	5

Now, the math gets a little more complex. What is the logarithm of 50? The answer is 1.698. What is the logarithm of 150? The answer is 2.176. What this simply means is that the number 10 taken to the 2.176 power is 150. These numbers are easily derived using a scientific type calculator.

Why are these 'base 10' logarithms useful? They form the basis of how engineers used to multiply numbers with slide rules. To multiply, we add logarithms, and to divide, we subtract logarithms. As an example, look at the problem

150 x 50.

We know that the answer is 7500.

Another way of working this problem is as follows:

log (50) + log (150) = log (answer)

or

1.698 + 2.176 = 3.874

And **(10) ^ 3.174 = 7500.**

The ability of logarithms to do division is also a feature of logarithms:

1000 / 50 = 20

Or

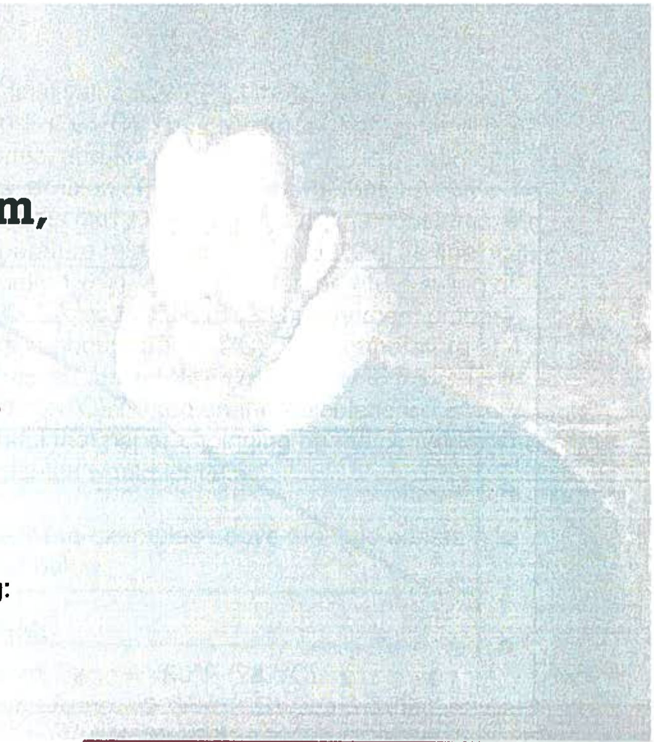
log (1000) – log (50) = log (answer)

3 - 1.698 = 1.302

Now, **10 ^1.302 = 20.**

So far, the fire investigator is scratching his or her head. Isn't it just easier to divide 1000 by 50 and obtain the answer 20? Probably so.

Note that so far, all the logarithms have been based on "base 10" math. The base 10 logarithms are useful in working with chemical equations involving pH (how acid or how *basic* is a chemical solution)?



An overview of the Napierian logarithm, the time constant, Fourier's Law, and the workings of nature

The Napierian logarithm

The Napierian logarithm is a logarithm which describes natural processes seen in the world. It differs from the 'base 10' logarithm. A common problem solved by the Napierian logarithm includes the following:

> *I had a capacitor (an electrical energy storage device). It had a voltage of 75 volts present, and a capacitance of 2200 uF (micro farads). A resistance of 10000 ohms was place across it. How long does it take the capacitor to discharge until the voltage is 20 volts?*

Fourier's Law of Heat Transfer

Inherent in any discussion regarding fires and the Napierian logarithm is a good understanding of Fourier's Law of Heat Transfer.

Fourier's Law (in simple fashion) is described by the following diagram:

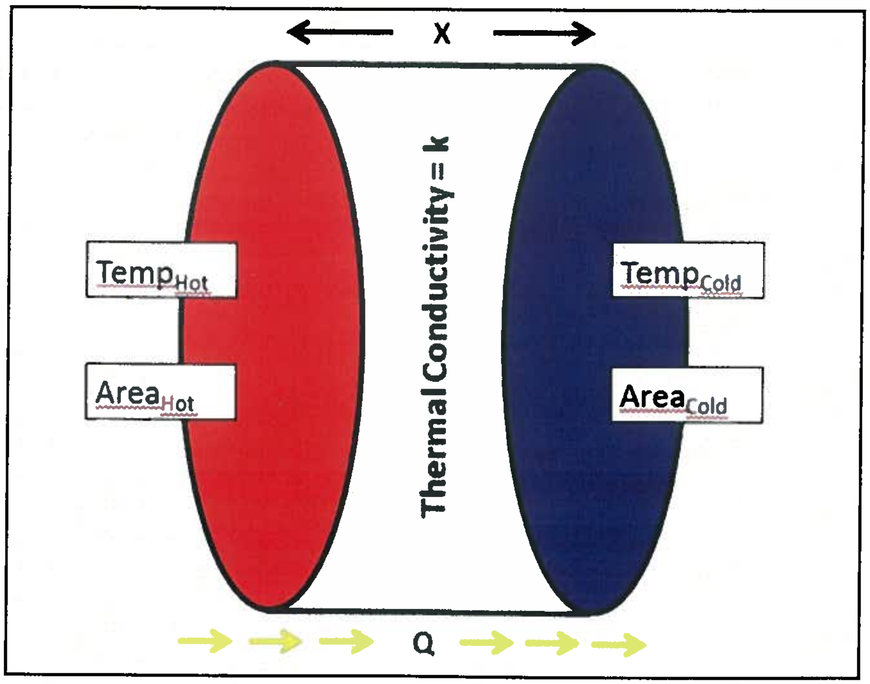


Illustration 1: Thermal Transfer from Hot to Cold Surface

The law simply states that the amount of thermal energy flowing from one surface to another surface (at any point in time) depends on:

The size of the surfaces (Area Hot and Cold, we assume are equal)
The distance between the two surfaces (Distance X)
The two temperatures of the surfaces (Temperature Thot and Tcold)
How well the medium (or intermediate material) between the two surfaces conducts heat. (Thermal conductivity k)

Fourier's Law is written as follows:

U (amount of energy transferred) = (k (thermal conductivity) x Area)) /

(Distance x (difference in temperature)

Or **U = (k A) / ((X (ΔT)),**

If we double the area of the surfaces, we double the flow rate of thermal energy. If we increase the distance, we lessen the flow rate. If we increase the thermal conductivity of the intermediate material (say change from fiberglass insulation to just air) we increase the rate of heat transfer.

For all of the three conditions above (area, distance, and thermal conductivity), Fourier's Heat Transfer Law works linearly (See chart 1); if we double the area, we double the flow rate. The Napierian twist to understanding this is that the flow rate of energy (as a function of time) decreases as time goes on (Exponential Function). Looking at the situation, we can state that as the temperatures between two surfaces grow closer in temperature (as one cools down and one increases in temperature), the rate of heat flow lessens.

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Chart 1: Linear (Straight Line) Function

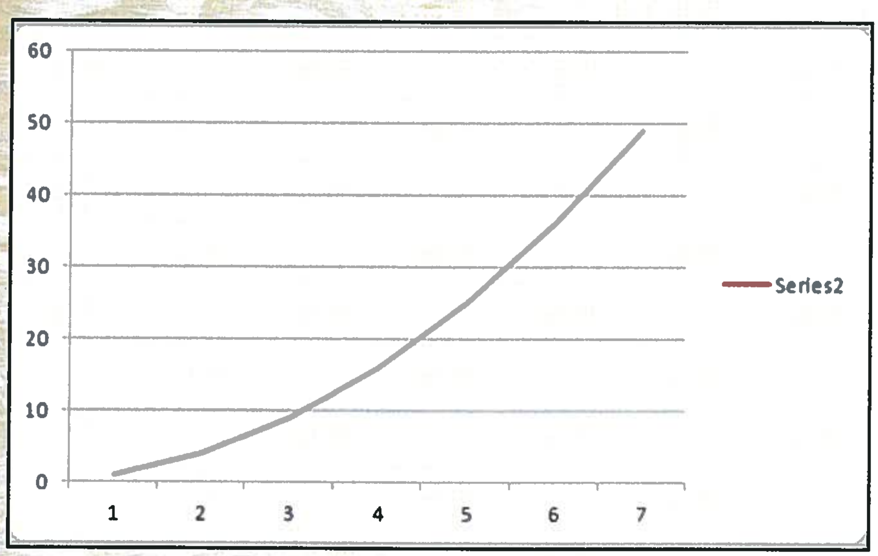


Chart 2: Napierian (Exponential) Function

Chart 3: Measurements of GMC Catalytic Converter Temperature (°C) Versus Time

Chart 4: Measurements of a Toyota Catalytic Converter Temperature (°C) Versus Time

The Time Constant

As we discuss the Napierian logarithm, we need to remember the concept of *time*. When speaking of fire investigations, time will often deal with flow rates of some quantity – be it energy, electricity, fuel, or air. Moreover, this rate of flow will often be associated with the Napierian logarithm.

The following graphs show the rise and fall of temperature on a catalytic converter on a GMC truck, as well as on a Toyota Tundra. Despite the differences in times and maximum temperatures, the graphs have similar appearances. The *time constant*, often referred to by the Greek letter τ *(tau)*, is defined as the time necessary to reach 63.2% of its excursion. On the downside, the time constant is the time necessary to drop to 36.8% of its excursion, or to drop *by* 63.2% of its excursion.

If one can determine what one (the quantity one) time constant is, then the system response can be calculated at any point in time. The equation for increases (rises) or falls (decays) of time constant systems is as follows:

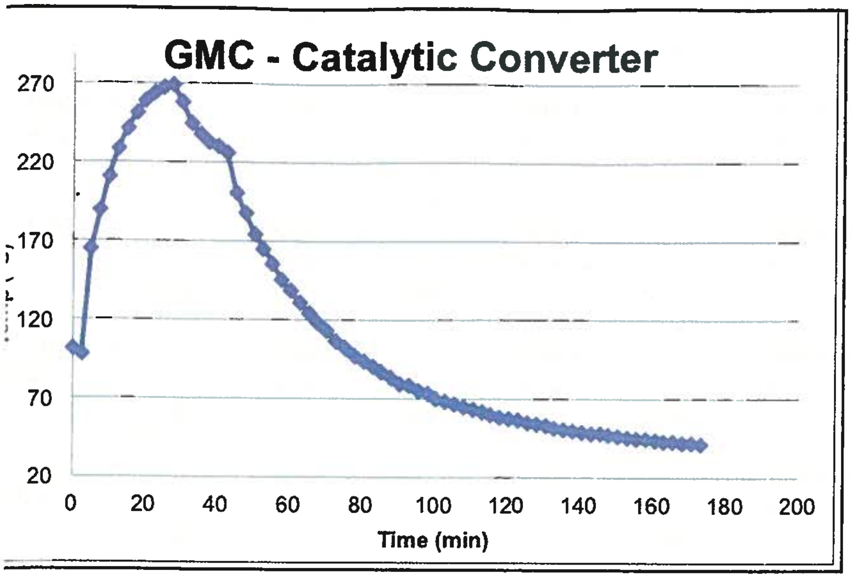
% change = $(1 - 1/e^{(t/\tau)})$ (x 100)

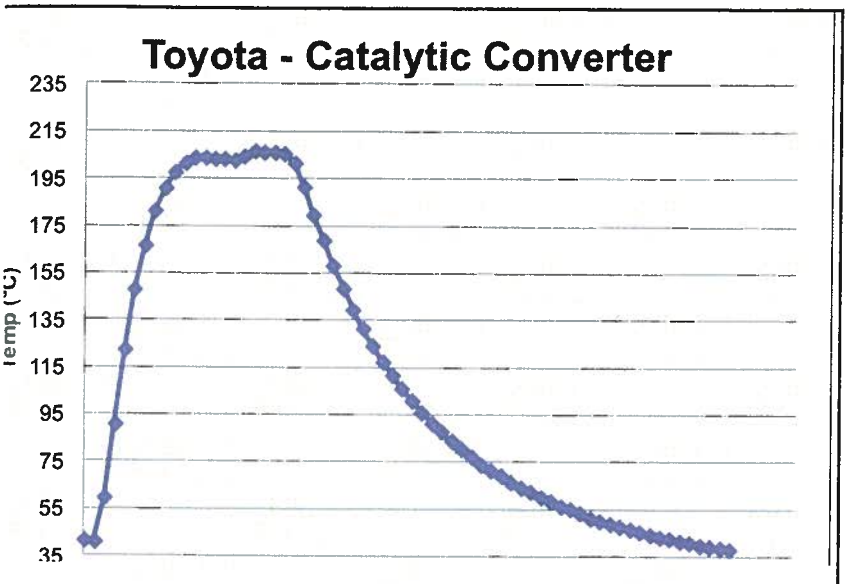
Where:
- **t = time in seconds**
- **τ = time constant tau**
- **e = Euler's number (2.718)**

The variable e is the 'base' for the Napierian or natural logarithm. (Euler's is pronounced like the former NFL team, the Houston *Oilers*).

Using this equation, we now can construct the following chart:

Number of Time Constants	Rise,%	Fall,%
1	63.2	36.8
2	86.4	13.6
3	95	5
4	98.1	1.9
5	99.3	.7





(The numbers are given as percentages of *displacement*. If a pressure change (displacement) is 100 psi (689 KPa), at one time constant it will have changed by 63.2 psi (435 KPa) as the system pressure is increasing. On the downside, at one time constant the pressure will have dropped by 63.2 %, or will have fallen to 36.8 % of its final pressure.

Engineers generally accept that when a process has risen or decayed to 5 time constants, there is no further change taking place.

So what does this chart mean? Pretend that a catalytic converter is being blamed for a car fire. Someone opined that wind inadvertently blew trash (Class A combustibles) under the car after being parked, trapping paper against the converter and igniting the fire. The *time constant* on an exemplar vehicle is found (by way of testing) to be 21 minutes. The vehicle reached a temperature of 1300°F (704°C), and the ambient was 75°F (24°C). The vehicle had been parked approximately 87 minutes when the fire started.

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The final value is 75°F (24°C). The initial value is 1300°F (704°C). Thus, we know that the time is 87 minutes, and the time constant τ is 21 minutes. Thus, division of 87 by 21 shows that 4.14 time constants had elapsed. At four time constants, the temperature has fallen to within 2% of its final value (actually 1.8%). Thus, for a temperature swing of (1300-75), or 1225°F (680°C), 2 percent of that figure is about 25°F (14°C). The temperature at 4 time constants is [75°F (24°C) + 27°F (-3°C)], or 102°F (21°C). Based on this calculation, it is very doubtful that paper impinging on the converter after 87 minutes would ignite.

For our two examples above the time constant is solved below.

GMC

High Temp = 460°F (237°C)
Low Temp = 90°F (32°C)
Therefore 460° - 90° = 370°F (205°C)
370°F * 0.632 = 234°F (130°C) and 460° - 234° = 226°F (107°C)

Looking up 226°F (107°C) on our graph our downward TC1 would be at 29 minutes.

Toyota

High Temp = 360°F (182°C)
Low Temp = 90° (32°C)
Therefore 360°F - 90°F = 270° (150°C)
270°F * 0.632 = 171°F (95°C) and 360°F - 171°F = 189°F (87°C)

Looking up 189°F (87°C) on our graph our downward TC1 would be at 37 minutes.

What Does This Mean?

The writers believe that the most important concept for the investigator to understand is that natural processes often are continually changing mathematically. By way of example, consider two adjacent rooms, each sharing a common wall. We know that the direction of heat flow in that heat travels from hot to cold. Further assume that one room has a temperature of 90°F (32°C), and one room has a temperature of 60°F (16°C). Heat will pass through the common wall from the warm room to the cool room. Initially, the rooms have a 30°F (16°C) difference and there will be a certain *rate* of initial heat transfer. The rate of heat transfer slows as the two rooms get closer in temperature, the rate of heat transfer decreases. Simply stated,

> The rate of heat transfer is determined by the differences in temperatures. As the temperatures become closer together, the rate of heat transfer decreases.

While we have used examples of heat, this same concept applies also in voltages and in pressures. A full gasoline tank that has a hole in the bottom of it will leak fuel faster when the tank is full compared to when it is almost empty.

These processes always have mathematical curves, which take on the form of a graph, as follows:

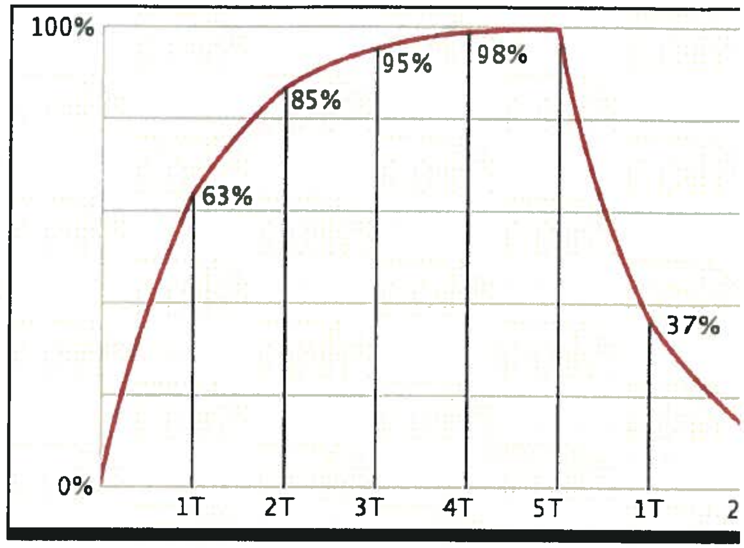

Chart 5: Showing Exponential Rise and Decay

The Problem Solved

We previously gave a problem that can be solved. After showing the background of the Napierian Theorem above, the reader should be able to solve the aforementioned problem. The mathematical solution to the capacitance problem above is below:

- **The time constant is defined by the equation R x C,** or

10,000 x 2200 E (-6) = 22 seconds for a resistor capacitor equation.

The initial value is 75 volts. The final value is 0 volts. We desire to know the time when the voltage has decayed to 20 volts. In other words, what time has elapsed when the voltage has decayed by 73.3 % of its initial value? From the above chart, we can see that for a decay of 73.3 %, the time will be between 1 and 2 time constants (22 and 44 seconds). The more exact value is found as 1.32 time constants. The math is below.

.733 = 1 – 1/ (e^t/22)

Or, **1.32 time constants.**

Or, **1.32 x 22 = 29.04 seconds.**

Measuring the Time Constant

Perhaps the biggest 'unknown' in this discussion is that of the time constant. The best way (in the writer's opinion) to get a time constant on a physical system is to build an exemplar fixture and measure the time constant. The *usual* parameter to be measured will be temperature, but voltages and pressures are also necessary sometimes. While instrumentation is beyond the scope of this paper, here is the way to determine a time constant.

If we measure an engine block's temperature as a function of time, we will get a graph like the ones shown above. Note that the initial (ambient) temperature is 90°F (32°C), and the final (high) temperature is 360°F (182°C). The total change is 270°F (150°C), found by subtracting the initial temperature from the final temperature. If we multiply the total change 270°F (132°C) by .632, we then have a value 171°F (87°C). That value 171°F (83°C) is added to the initial temperature of 90°F (32°C). When that temperature is reached (90 + 171), we look at how much time has elapsed. That gives us our time constant.

Pitfalls

There are (as usual) certain cautions the investigator must be aware of when trying to measure time constants. These *caveats* are as follows:

1.) For catalytic converter testing, if a cooling fan or cooling system is introduced into the system after the testing has started, that can change the results. After an engine is powered off, it often will rise in temperature as heat flux temporarily accumulates and is not being sent to the atmosphere by means of a fan or radiator.

2.) If ambient temperature suddenly changes (ie, a cool front comes in), the results will be changed.

3.) When measuring cooling rates, the concept of time constant must be modified for materials during phase change. An example of phase change would be, when ice melts, the water changes phase from a solid to a liquid, or when water boils the phase changes from a liquid to a gas.

Summation

This paper has presented the fire investigator with some of the basic mathematical concepts that deal with fire science. While the math can be unwieldy for someone who have not used math books in many years, the concepts behind the math (and science) are not that difficult to grasp. More importantly, these mathematical concepts help to explain how many things in nature, including heat transfer, work in the real world.

While testing may be easier this paper gives the reader the ability to use math to solve gradient (Time based) problems such as heat transfer, decreases in air pressure and voltage decrease using time as a variable. These concepts will come in handy when a fire investigator is required to determine temperature after a cool down period, or water flow from a sprinkler reservoir after the sprinkler head was activated.